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A DISPLACEMENT BOUNDING PRINCIPLE IN FINITE PLASTICITY

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Abstract—A deformation bounding principle is developed for finitely deforming, rigid, perfectly-plastic structures exhibiting geometric stability. The problem is formulated in terms of Kirchhoff's stress tensor and Green's strain tensor and the principle of virtual work and Drucker's Postulate is applied.

The principle is used to find a bound on the response of a cylindrical shell under internal pressure and the bound is compared to the exact results also obtained herein.

INTRODUCTION

THE effect of geometric changes on the load carrying capacity of structures is an important corollary to limit analysis. In limit analysis, both work-hardening and geometric effects are neglected. Experiments have shown, however, that some structures may effectively sustain loads considerably in excess of the limit load, whereas other structures collapse before the limit load is attained. Since the inclusion of work-hardening in any constitutive model strengthens the structure, the questions of structural stability and effective load carrying capacity may only be resolved by including in analysis the effect of geometric changes on the structural response to load histories.

The question of uniqueness and stability of a rigid-plastic continuum at incipient deformation was first discussed by Hill [1], who formulated the problem in terms of the nominal stress tensor. If the structure proves to be stable, it is important to have a measure of the post yield load-deformation behavior. Two approaches to solving problems of this class have been used. In the first, formulated by Onat [2], the equilibrium equations are written in a rate form involving both stresses and displacements and the complete solution of the resulting initial value problem is found in terms of a time-like parameter. This method was used by Onat and Shu [3, 4] to find the load-deflection behavior of circular arches and by Jones [5, 6] for a number of static and dynamic problems. In the second approach, a sequence of limit analysis problems is solved. At each step, a velocity field is determined; and thus the geometry of the resulting configuration may be used to determine a new load carrying capacity. Coon and Gill [7] have used this method to evaluate how geometric changes affect the limit load of cylindrical shells. Circular plates were discussed in this sense by Lance and Onat [8].

The number of problems which have been solved taking geometric changes into account appears to have been limited by the lack of suitable numerical algorithms and methods for obtaining rational approximations. Such methods have been used successfully in elasticity and the theory of perfect plasticity at small deformations. Thus, for example, the limit analysis theorems in plasticity may be applied to find upper and lower bounds on the collapse load. Furthermore, they form the basis for finding numerical solutions using mathematical programming techniques [9]. Similarly, bounding principles have been developed for a number of models of material response. Although bounding principles differ from strict minimum principles in that the bound cannot be brought arbitrarily close to the exact solution, they have produced useful results with a minimum of computational difficulty.

All of the foregoing methods including the bounding principles of structural analysis are based on the principle of virtual work and the constitutive relations of the material. In each case, either an equilibrium stress field, or a displacement or velocity field must be constructed, and, through the constitutive equations and the principle of virtual work, an inequality on some physically meaningful quantity such as the collapse load is obtained.

One would like to use the same approach to construct bounding principles for a class of problems where geometric effects are of importance. However, one immediately finds that the stress and displacement fields are coupled in the equilibrium equation and thus one should not expect to be able to construct principles using solely stress fields or velocity fields. Guided by these considerations, we shall proceed to develop a bounding theorem for a class of structures exhibiting geometric stability. In form, the bounding theorem is similar to the upper bound theorem of limit analysis. We shall assume that Drucker's postulate of material stability is satisfied. To illustrate the application of the bounding principle, we shall consider as an example the post-yield response of a simply supported cylindrical shell under internal pressure. The exact solution of the problem will be found and the results compared with bounds on the displacement constructed using the bounding principle.

PRINCIPLE OF VIRTUAL WORK

When geometric changes are taken into account, it becomes necessary to make a distinction between the original and the deformed configuration of the body. Consequently, two different approaches may be taken. The fundamental field variables, stresses and strains, may be referred either to a fixed reference, for example the undeformed configuration of the continuum, or they may be identified with the position of the material elements of the body in its deformed state (see, for example, Prager [10] or Fung [11]). In the theory of plasticity, the constitutive equations must be expressed as a relation between stress rates and strain rates. If stress rates and strain rates are to have physical meaning, they must vanish identically when the body is subjected to a rigid body rotation. This restriction leads to more complicated expressions if the second approach is used and for this reason all quantities shall henceforth be referred to the undeformed configuration of the continuum.

Using this approach, the appropriate definitions for stress and strain are the Kirchhoff stress tensor and Green's strain tensor [11], respectively. The equilibrium equation then is†

$$
\begin{aligned}
(S_{ji}+S_{jk}u_{i,k})_{,j} &= 0 \quad \text{in } V \\
(S_{ji}+S_{jk}u_{i,k})n_j &= T_{j\to i} \quad \text{on } S \\
S_{ji} &= S_{ij}
\end{aligned}
\tag{1}
$$

where u_i is the deformation measured from the undeformed state, n_j is the unit outward normal to the undeformed surface S and T_i are the surface tractions per unit area of the undeformed surface. V and S are taken to be the original volume and surface respectively and the independent variables x_i locate the original position of the material elements. The corresponding strain tensor is

$$E_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}). \tag{2}$$

The rate formulation follows very simply; the equilibrium rate equations and strain rate tensor being given by

$$(\dot{S}_{ji}+\dot{S}_{jk}u_{i,k}+S_{jk}\dot{u}_{i,k})_{,j} = 0 \tag{3}$$

$$\dot{E}_{ij} = \tfrac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}+\dot{u}_{k,i}u_{k,j}+u_{k,i}\dot{u}_{k,j}). \tag{4}$$

(We assume a quasi-static process; thus inertia terms in equation (3) are neglected.)

The principle of virtual work, as has been mentioned before, plays a central role in the derivation of approximate theories for one- and two-dimensional continua and also in the formulation of bounding theorems. We recall that in the linear, small deformation theory it may be stated as

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \int_S T_i\delta u_i\,\mathrm{d}S \tag{5}$$

where σ_{ij} is any arbitrary stress field satisfying the equilibrium equations and stress boundary conditions and $\delta\varepsilon_{ij}$ is the strain field which is derivable from any varied displacement field δu_i satisfying the displacement boundary conditions. In the theory of plasticity it is customary to speak of strain rates rather than strain increments; therefore we shall write the equation of virtual work in an equivalent form

$$\int_V \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}V = \int_S T_i\dot{u}_i\,\mathrm{d}S \tag{6}$$

with σ_{ij}, $\dot{\varepsilon}_{ij}$ and $\dot{u}_i$ subject to the same conditions as before. In the theory of finite deformations outlined above the form of the principle of virtual work remains the same

$$\int_V S_{ij}\dot{E}_{ij}\,\mathrm{d}V = \int_S T_i\dot{u}_i\,\mathrm{d}S. \tag{7}$$

However, the stress and displacement fields are no longer independent. Rather, we must choose a stress field S_{ij} and a displacement field u_i together in such a way that they satisfy

† We employ customary indicial notation. A repeated subscript implies summation over that index, $(\ )_{,j}$ differentiation with respect to x_j and $(\dot{\ })$ differentiation with respect to "time".

the equilibrium equation (1) and all boundary conditions. Then, independently, we may select any velocity field $\dot{u}_i$ satisfying velocity boundary conditions. The strain rate tensor is then determined from u_i and $\dot{u}_i$ by equation (4). A proof of equation (7) follows.

The left hand side of equation (7) is equivalent to

$$\int_V S_{ij}\dot{E}_{ij}\,dV = \int_V (S_{ji}+S_{jk}u_{i,k})\dot{u}_{i,j}\,dV \tag{8}$$

under the condition that S_{ij} be symmetric and $\dot{E}_{ij}$ satisfy the equations of compatibility. Integrating equation (8) by parts, we obtain

$$\int_V S_{ij}\dot{E}_{ij}\,dV = \int_V (S_{ji}+S_{jk}u_{i,k})_{,j}\dot{u}_i\,dV + \int_S (S_{ji}+S_{jk}u_{i,k})\dot{u}_i n_j\,dS. \tag{9}$$

The volume integral in equation (9) is identically zero if S_{ij} and u_i satisfy the equation of equilibrium. Under the same conditions, the surface integral reduces to the form given by equation (7).

It should be noted that the equation of virtual work was derived without making any assumptions regarding material behavior. Furthermore, the result is still valid if the problem is formulated in terms of properly defined generalized stresses and strains.

CONSTITUTIVE EQUATIONS

Green and Naghdi [12] formulated a general theory for elastic–plastic continua making full use of the restrictions imposed by the principles of invariance and laws of thermodynamics. In particular, they assumed that plastic behavior was characterized by the existence of a yield surface: if and only if the stress lies on the yield surface and the rate of stress vector is directed towards the exterior of the yield surface, the plastic component of the strain rate is non-zero and loading occurs. Using this model and assuming an isothermal continuum, they derived the restriction

$$(S_{ij}-S_{ij}^0)\dot{E}_{ij} > 0 \tag{10}$$

where S_{ij} and E_{ij} are associated through some constitutive equation and S_{ij}^0 may be called the "thermodynamic reference stress". Phillips and Eisenberg [13] show that this reference stress must always be within the yield surface. There are two further assumptions which are customarily made: convexity of the yield surface and normality of the strain rate vector to it. The former is supported by physical evidence: the latter is a consequence of Drucker's Postulate of stability in the small

$$\dot{S}_{ij}\dot{E}_{ij} \geq 0. \tag{11}$$

(By stability we refer here to material stability, not geometric stability). It has been shown that Drucker's Postulate is not essential to construct a consistent axiomatic theory [13]; whether or not it is essential for a physically valid theory is still an open question. However, it has proved useful in many applications and insures material stability in the infinitesimal theory without precluding unstable configurations in the finite theory. Henceforth we shall assume that Drucker's Postulate does apply. Then inequality (11) requires

$$(S_{ij}-S_{ij}^*)\dot{E}_{ij} \geq 0 \tag{12}$$

where S_{ij}^* is some stress point within the yield surface and $\dot{E}_{ij}$ is the rate of strain normal to the loading surface at the plastic stress point S_{ij}.

BOUNDING PRINCIPLE

At this point we shall specialize further to consider a class of problems which exhibit both stability and uniqueness. We also assume that, though geometrical effects are of significance, the strains are small enough that we are justified in assuming a yield condition of the form

$$f(S_{ij}) = k \tag{13}$$

where k is a constant. The plausibility of assumption (13) may be based on experimental evidence such as "true" stress–strain behavior of ductile materials in a uniaxial test. On the other hand (13) may merely be accepted as a hypothesis for the as-yet untested consequences of the theory developed below. It should be noted that this function does not describe exactly a perfectly plastic material since the Kirchhoff stress does not equal the actual stress on the material element. Structures which satisfy these restrictions are plates loaded such that the principal membrane forces are positive and, presumably, most shells under internal pressure. Assuming a perfectly plastic response, we note that the response of such structures falls into three stages: initial flow, an intermediate stage and, finally, the membrane state. The first stage may be found by the usual methods of limit analysis. The exact solution to the second stage is in general a difficult problem and thus approximate methods are desirable. The principle of virtual work was used by Onat [14] to determine approximate load–deflection curves for circular plates and by Sawczuk [15] for the rectangular plates. In the last stage, bending moments are identically zero and the structure acts as a membrane.

As in limit analysis we take recourse to the principle of virtual work. Consider a structure whose collapse load in the small deflection theory is T_i and consider the same structure on further loading such that the tractions are λT_i, $\lambda > 1$. (We are here making use of stability by assuming the structure can support λT_i.) Given λ, assume some displacement field u_i such that we can find a stress field S_{ij} nowhere violating yield, which in conjunction with u_i will satisfy the equilibrium equation and boundary condition (1). Consider independently a velocity field $\dot{u}_i$, which together with u_i serves to define $\dot{E}_{ij}$. Then, by Drucker's Postulate (12)

$$D(\dot{E}_{ij}) \geq S_{ij}\dot{E}_{ij}. \tag{14}$$

Taking the volume integral of inequality (14) and making use of equation (7), we obtain

$$\int_V D(\dot{E}_{ij})\,\mathrm{d}V \geq \lambda \int_S T_i\dot{u}_i\,\mathrm{d}S. \tag{15}$$

This inequality is exactly the familiar one of limit analysis. However, we have the additional restriction that a suitable stress field must be found. This is to be expected since stress and kinematic terms are coupled in this case whereas they are independent in the linear case. Although the problem so far has been formulated in terms of a three-dimensional continuum, all results will hold equally well if generalized stresses and strains are used.

SIMPLY SUPPORTED CYLINDRICAL SHELL

As an illustration, let us consider the response of a rigid perfectly plastic cylindrical shell, pinned at the ends, under internal pressure. This problem was first considered by Duszek [16]; however, on closer examination it may be seen that the strain rates she obtains violate the flow rule. We shall study the post-yield behavior of the structure by means of the method used by Onat [2]. Thus, we shall obtain solutions to a rate problem, with the initial values found by the usual methods of limit analysis. Finally, we shall compare the exact results with bounds on the displacements obtained using the bounding principle developed herein.

Consider a cylindrical shell of length $2L$, radius A and thickness $2H$ under an internal pressure P as illustrated in Fig. 1. Since the geometry and loading are axi-symmetrical, the only nonvanishing stresses acting on any portion of the shell are S_{rr}, S_{xx}, $S_{\Phi\Phi}$ and S_{xr} in

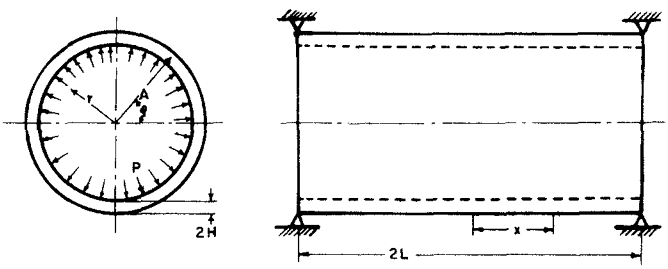

FIG. 1. Cylindrical shell.

cylindrical coordinates. Furthermore, since the shell is assumed thin, S_{rr} is small compared to the other stress components. Generalized stresses, Fig. 2, may now be defined by taking the average and the first moment of the stress components over the thickness:

$$N_x = \int_{-H}^{H} S_{xx}\,\mathrm{d}\rho \qquad N_\Phi = \int_{-H}^{H} S_{\Phi\Phi}\,\mathrm{d}\rho \qquad Q_x = \int_{-H}^{H} S_{xr}\,\mathrm{d}\rho$$
$$M_x = \int_{-H}^{H} S_{xx}\rho\,\mathrm{d}\rho \tag{16}$$

where ρ is measured from the middle surface of the shell.

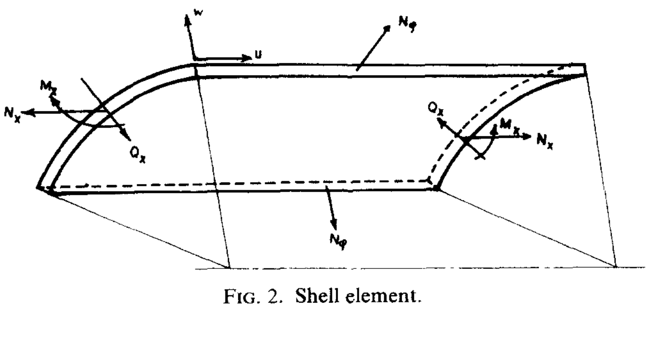

FIG. 2. Shell element.

The displacements are W and U in the radial and axial direction, respectively. We assume that the shell thickness is small, i.e.

$$H \ll L \tag{17}$$

that

$$W = O(H) \qquad |U| \ll |W| \tag{18}$$

that the slope of W is small and that Kirchhoff's hypotheses hold. Thus we assume that plane sections normal to the central surface remain plane and normal; and therefore the shear Q_x does no work and may be considered a reaction.

The equilibrium equations may be derived directly from the full, three-dimensional field equations or the principle of virtual work may be applied. The first approach is used by Fung [11] to derive the equilibrium equations for the finite deflection of plates. On non-dimensionalizing in the following manner†

$$n_x = \frac{N_x}{2\sigma_0 H} \quad n_\Phi = \frac{N_\Phi}{2\sigma_0 H} \quad m_x = \frac{M_x}{\sigma_0 H^2} \quad p = \frac{AP}{2\sigma_0 H}$$
$$x = \frac{X}{L} \quad w = \frac{W}{A} \quad u = \frac{U}{L} \quad R = \frac{A}{H} \quad \alpha = \frac{L^2}{AH} \quad v = \frac{A^2}{L^2} \tag{19}$$

the generalized strains become

$$\lambda_x = u' + \tfrac{1}{2}vw'^2 \qquad \lambda_\Phi = w$$
$$\kappa_x = \frac{w''}{2\alpha} \qquad \kappa_\Phi = 0 \tag{20}$$

and the equilibrium equations

$$m_x'' - 2Rn_x w'' + 2\alpha n_\Phi - 2\alpha p = 0$$
$$n_x' = 0. \tag{21}$$

σ_0 is the yield stress in tension and primes denote differentiation with respect to the axial coordinate x.

For the particular problem under consideration, the boundary conditions are

$$m_x(1) = u(1) = w(1) = 0$$

and by symmetry (22)

$$m_x'(0) = u(0) = 0.$$

We assume a yield condition circumscribed on the exact Tresca yield condition as shown in Fig. 3, which has also been used by Drucker and Shield [17].

The yield surface is formed of six surfaces defined by the relations

$$\text{I. } n_\Phi = \pm 1 \qquad \text{II. } n_\Phi - n_x = \pm 1 \qquad \text{III. } \pm m_x + n_x^2 = 1. \tag{23}$$

† We follow the notation used by Duszek [16].

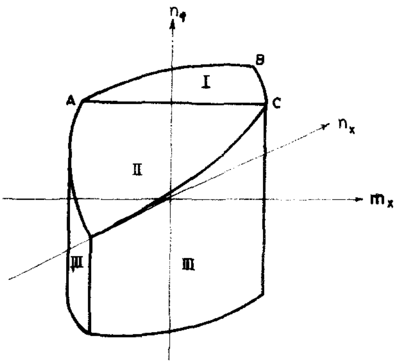

FIG. 3. Yield condition.

LIMIT ANALYSIS

Under monotonically increasing loading, incipient deformation first occurs at the yield pressure given by limit analysis. The yield pressure, stress and velocity fields must satisfy the linearized equilibrium equations

$$\begin{aligned} m_x'' + 2\alpha(n_\Phi - p) &= 0 \\ n_x' &= 0 \end{aligned} \tag{24}$$

the yield condition (23) and the flow rule for the linearized strain rates. The stress field lies entirely on the ridge AC of the yield surface, Fig. 3, with $m_x \leq 0$. A hinge forms in the shell at $x = 0$, the point corresponding to the corner A on the yield surface. The complete solution of the limit analysis problem is

$$\begin{aligned} p &= 1 + \frac{1}{\alpha} \\ m_x &= x^2 - 1 \\ n_\Phi &= 1 \qquad n_x = 0 \\ \dot{w} &= \dot{w}_0(1-x) \qquad \dot{w}_0 > 0. \end{aligned} \tag{25}$$

POST YIELD BEHAVIOR

In order to determine the behavior of the shell for loads greater than the limit load, we must recast the equilibrium equation (21) into rate form

$$\begin{aligned} \dot{m}_x'' - 2R\dot{n}_x w'' - 2Rn_x\dot{w}'' + 2\alpha\dot{n}_\Phi - 2\alpha\dot{p} &= 0 \\ \dot{n}_x &= \dot{C}. \end{aligned} \tag{26}$$

It is easy to see from equation (26) that initially $\dot{p} = 0$, or letting the central deflection $w(0)$ be our time-like parameter for this discussion only

$$\frac{\mathrm{d}p}{\mathrm{d}w(0)} = 0. \tag{27}$$

Thus the knowledge of the stress field at yield gives no information about the geometric stability of the shell on further loading.

On physical grounds we assume that $\dot{n}_x > 0$ (as the shell expands radially, it must stretch in the axial direction). We note that continued flow is possible only for stresses in region I of the yield surface or on the edges formed by the intersection of surfaces I and III. Thus

$$\dot{n}_\Phi = 0.$$

Also, an inspection of the stress field (25) shows that a violation of yield may be expected initially at $x = 0$ if m_x and p are both scaled in accordance with equation (24). Close to the support $x = 1$, where $m_x = 0$, the scaled solution is still valid. Therefore the point at which $m_x = n_x^2 - 1$ (the edge given by the intersection of I with III) moves toward the support. We shall denote this point by $x = \xi$. We make the hypothesis that for $x > \xi$, the radial displacement is linear; for $x < \xi$ the axial curvature is non-zero. In this latter region there are two possibilities to be considered: (1) the stress field lies entirely on the corner as was assumed by Duszek or (2) it is in region I. As noted before, the velocity field which follows from the first assumption and the equilibrium equations (26) violates the flow rule; the second assumption must therefore apply. Furthermore, by the flow rule and the condition $\dot{u}(0) = 0$, we obtain an additional boundary condition

$$\dot{w}'(0) = 0 \tag{28}$$

unless $n_x = 0$.

There are two possibilities to be considered at $x = \xi$; the velocity field and all appropriate derivatives are continuous or a travelling hinge forms. The first assumption requires $\dot{w}'$ to be continuous, which is impossible in view of the boundary condition (28). Thus we shall assume that a travelling hinge forms at $x = \xi$, and we may let ξ be our parametric time. (We assume ξ is monotonically increasing.) Then, the following relations are a consequence of the flow rule

$$0 \le x < \xi, \qquad \xi < x \le 1 \qquad \text{region I}$$

$$\dot{w}'' = 0 \qquad \dot{w} > 0 \tag{29}$$

$$\dot{u}' = -\nu w' \dot{w}'$$

and at $x = \xi$

$$\dot{\lambda}_x = -2n_x \dot{\kappa}_x. \tag{30}$$

The rate boundary conditions are obtained directly from equation (22) and consequently we obtain as a solution to the rate problem

$$0 \le x < \xi \qquad \begin{aligned} \dot{m}_x &= 2R\dot{n}_x w + \alpha \dot{p} x^2 + \dot{B} \\ \dot{n}_x &= \dot{C} \\ \dot{w} &= \dot{D} \\ \dot{u} &= 0 \end{aligned} \tag{31}$$

$$\xi < x \le 1 \qquad \begin{aligned} \dot{m}_x &= \alpha \dot{p}(x^2 - 1) + \dot{J}(1-x) \\ \dot{w} &= \dot{E}(1-x) \\ \dot{u} &= \nu E \dot{E}(1-x) \end{aligned} \tag{32}$$

with six unknowns: $\dot{B}$, $\dot{C}$, $\dot{D}$, $\dot{E}$, $\dot{J}$ and $\dot{p}$.

In writing the equilibrium equation (21) we assumed continuity in m_x' and w'. Continuity in m_x and w across a moving hinge is automatically satisfied by any rate field. However, continuity in the slopes requires that $\dot{m}_x$ and $\dot{w}$ be continuous since continuity of m_x across a moving hinge requires that

$$[\dot{m}_x]+\dot{\xi}[m_x'] = 0 \tag{33}$$

where [ ] denotes the jump in the quantity enclosed in brackets. Thus, continuity of $\dot{m}_x$ follows from the assumption that $[m_x'] = 0$.

Two further restrictions are found by requiring that at $x = \xi$

$$m_x = n_x^2-1 \tag{34}$$

and since m_x assumes a minimum there

$$m_x' = 0. \tag{35}$$

A fifth condition is given by equation (30) at $x = \xi$, which is equivalent to

$$\alpha[\dot{u}] = -n_x[\dot{w}']. \tag{36}$$

The sixth condition is found by requiring that the equilibrium rate equation hold at $x = \xi$

$$[\dot{m}_x']-2Rn_x[\dot{w}'] = 0 \tag{37}$$

or equivalently, that the equilibrium equation (21) be satisfied on the interval $[0, \xi)$.

Subject to the above continuity conditions and the initial conditions (25) we obtain

$$\begin{aligned}
&\alpha(p-1) = \frac{1}{1-\xi}, \qquad n_x = \sqrt{\xi}, \qquad n_\Phi = 1 \\
&0 \le x < \xi \qquad u = \frac{1}{2R\alpha}\frac{x}{1-x}, \qquad Rw = 2\sqrt{x}-\sqrt{\xi}+\ln\frac{(1+\sqrt{\xi})(1-\sqrt{x})}{(1-\sqrt{\xi})(1+\sqrt{x})} \\
&m_x = -2\xi+4\sqrt{(\xi x)}+\frac{x^2-1}{1-\xi}+2\sqrt{\xi}\ln\frac{(1+\sqrt{\xi})(1-\sqrt{x})}{(1-\sqrt{\xi})(1+\sqrt{x})} \\
&\xi < x \le 1 \qquad u = \frac{\xi(1-x)}{2R\alpha(1-\xi)^2}, \qquad Rw = \frac{\sqrt{\xi}}{1-\xi}(1-x) \\
&m_x = \frac{1}{1-\xi}[x^2-1+2\xi(1-x)].
\end{aligned} \tag{38}$$

The above solution is valid until $\xi = \frac{1}{3}$. On further loading, the solution violates the yield condition at ξ^-. We shall now assume that the hinge splits into two parts, one travelling toward the support, the other toward the center. We further assume that the region between the two hinges remains on the edge AB, since

$$-\left[\frac{(\dot{p}-1)}{nx}\right],$$

a quantity heretofore positive and corresponding to the curvature rate for a stress field on AB, is zero at $\xi = \frac{1}{3}$.

We denote the hinge travelling to the center by η and the one travelling toward the support by ξ. m_x is a constant in the region (η, ξ) and the equilibrium equation may be integrated directly there to give the displacement w. Similarly, in the region $(\xi, 1]$, the displacement w is linear and equation (21) may be integrated directly for m_x. In both regions, the axial displacement rate follows directly from the flow rule. In the third region $[0, \eta)$, $\dot{w}$ must be a constant in view of the boundary condition (28). Thus, the equilibrium rate equation (26) may be integrated directly in $[0, \eta)$. We obtain

$$\begin{aligned}
0 \le x < \eta \qquad & \dot{m}_x = 2R\dot{n}_x w + \alpha \dot{p} x^2 + \dot{B} \\
& \dot{w} = \dot{C} \qquad \dot{u} = 0 \\
& \dot{n}_x = \dot{D} \\
\eta < x < \xi \qquad & m_x = n_x^2 - 1 \\
& w = \frac{-\alpha(p-1)}{2Rn_x} x^2 + Ex + F \\
& \dot{u} = \frac{-\alpha}{6R}\left[\frac{(p\dot{-}1)^2}{n_x^2}\right] x^3 + \frac{n_x}{R}\left[\frac{(\dot{p}-1)}{n_x}\right] x + \dot{G} \\
\xi < x \le 1 \qquad & m_x = \alpha(p-1)(x^2-1) + J(1-x) \\
& w = K(1-x) \\
& u = \frac{v}{2} K^2 (1-x)
\end{aligned} \tag{39}$$

subject to

$$\begin{aligned}
m_x'(\xi) = m_x'(\eta) &= 0 \\
[\dot{m}_x(\eta)] = [\dot{w}(\eta)] &= 0 \\
[w(\xi)] = [w'(\xi)] = [m_x(\xi)] &= 0 \\
[\dot{m}_x'(\eta)] - 2Rn_x[\dot{w}'(\eta)] &= 0 \\
[\dot{u}(\eta)] &= \frac{-n_x}{\alpha}[\dot{w}'(\eta)] \\
[\dot{u}(\xi)] &= \frac{-n_x}{\alpha}[\dot{w}'(\xi)].
\end{aligned} \tag{40}$$

The solution to the problem for $\xi > \frac{1}{3}$, $(\eta < \frac{1}{3})$ is

$$
\begin{aligned}
0 \le x < \eta \qquad & \dot{m}_x = 2R\dot{n}_x[w(x)-w(\eta)]+\alpha\dot{p}(x^2-\eta^2)+2n_x\dot{n}_x \\
\eta < x < \xi \qquad & m_x = -\alpha(p-1)(1-\xi)^2 \\
& w = -\frac{\alpha(p-1)}{2Rn_x}(x^2+\xi^2-2\xi) \\
& \dot{u} = -\frac{\alpha}{6R}\left[\frac{(p \dot{-} 1)^2}{n_x^2}\right](x^3-\eta^3)+\frac{n_x}{R}\left[\frac{(p \dot{-} 1)}{n_x}\right]x \\
\xi < x \le 1 \qquad & m_x = \alpha(p-1)(x^2-1+2\xi-2\xi x) \\
& w = \frac{\alpha(p-1)\xi(1-x)}{Rn_x} \\
& u = \frac{\alpha(p-1)^2\xi^2(1-x)}{2Rn_x^2} \\
& n_x = \{1-\alpha(p-1)(1-\xi)^2\}^{\frac{1}{2}} \\
\sqrt{\eta(1-\eta)} = & \frac{n_x}{\alpha(p-1)}
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
& \alpha\dot{p}\{2-\alpha(p-1)(1-\xi)^2\}[3\xi\{1-\alpha(p-1)(1-\xi)\}+\alpha(p-1)(\eta^3-\xi^3)] \\
& \quad +\alpha(p-1)[6\{1-\alpha(p-1)(1-\xi)\}^2-2\alpha^2(p-1)^2(1-\xi)(\eta^3-\xi^3)] = 0.
\end{aligned}
\tag{42}
$$

Equation (42) was integrated numerically and the results are shown in Figs. 4–6. This solution is valid until $\alpha(p-1) \simeq 4{\cdot}5$. At this point

$$m_x(0) = 1-n_x^2$$

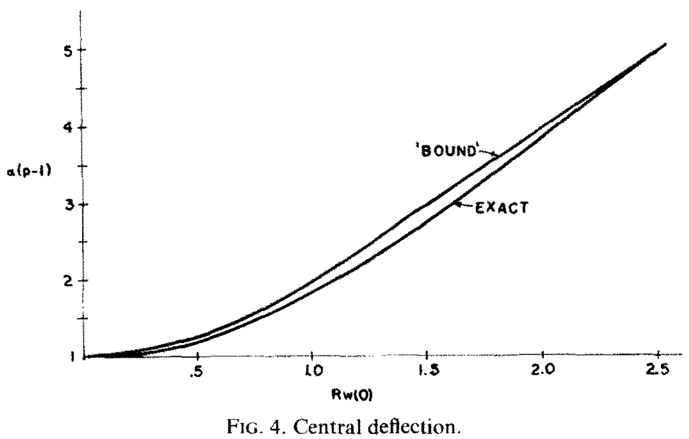

FIG. 4. Central deflection.

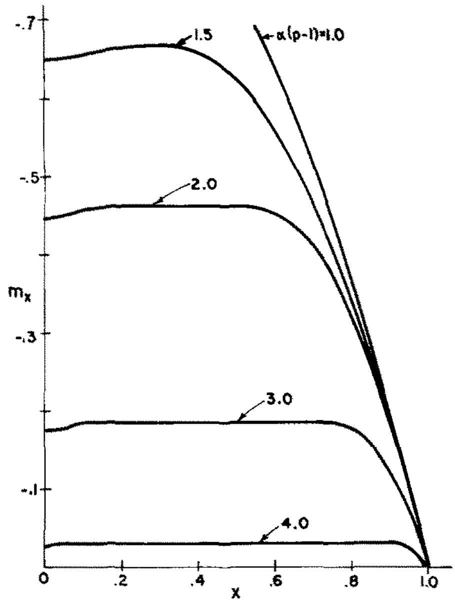

FIG. 5. Bending moment.

and another hinge forms at the center until the membrane solution

$$w = \frac{\alpha(p-1)}{2R}(1-x^2)$$

$$m_x = 0 \qquad n_x = 1 \qquad n_\Phi = 1$$

is attained at $\alpha(p-1) \approx 4{\cdot}75$.

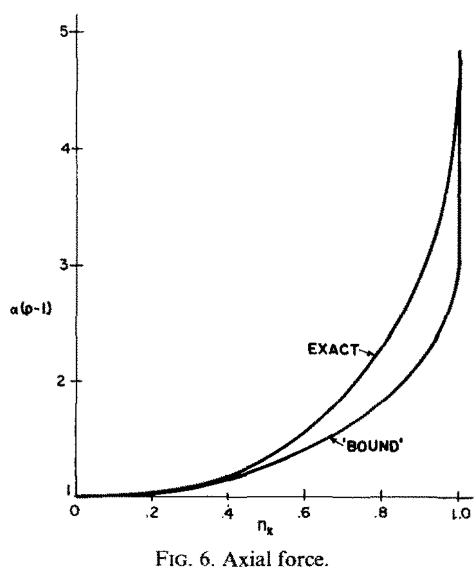

FIG. 6. Axial force.

The exact solution to the rate problem is difficult to obtain even for a simple configuration such as the one considered above. For purposes of comparison, we shall now consider the approximate results that may easily be obtained using the bounding principle given by equation (15).

BOUND

The first step is to construct an approximate displacement field and a velocity field. The curvature κ or w'' in some central region must be nonzero so that we may construct a stress field nowhere violating yield. Thus by a Taylor expansion, we assume

$$\begin{array}{ll} 0 \leq x < \xi & w = w_1(2\xi - \xi^2 - x^2) \\ & \dot{w} = \dot{w}_2(2\xi - \xi^2 - x^2) \\ \xi < x \leq 1 & w = 2w_1\xi(1-x) \\ & \dot{w} = 2\dot{w}_2\xi(1-x) \end{array} \tag{43}$$

where w_1 and $\dot{w}_2$ are arbitrary and we have satisfied continuity in w' and $\dot{w}'$. The dissipation D is then given by

$$D = m_x\dot{\kappa}_x + n_x\dot{\lambda}_x + n_\Phi\dot{\lambda}_\Phi \tag{44}$$

and by equation (23) and the assumed velocity field

$$D = -\dot{\kappa}_x(1+n_x^2) + \dot{\lambda}_\Phi \tag{45}$$

where

$$\begin{array}{lll} 0 \leq x < \xi & \dot{\kappa}_x = -\dfrac{\dot{w}_2}{\alpha} & \dot{\lambda}_\Phi = \dot{w}_2(2\xi - \xi^2 - x^2) \\ \xi < x \leq 1 & \dot{\kappa}_x = 0 & \dot{\lambda}_\Phi = 2\dot{w}_2(1-x)\xi. \end{array} \tag{46}$$

Finally, by equation (15)

$$\int_0^1 D(x)\,\mathrm{d}x \geq \int_0^1 P\dot{w}\,\mathrm{d}x \tag{47}$$

and substituting equations (45) and (46) into (47), we obtain the inequality

$$\alpha(p-1) \leq \frac{(1+n_x^2)}{(1-\xi^2/3)}. \tag{48}$$

It now remains to find a stress field satisfying equations (21) and (22) and nowhere violating yield. On the interval $[0, \xi)$ we obtain, assuming $n_\Phi = 1$

$$m_x'' = -4Rn_xw_1 + 2\alpha(p-1). \tag{49}$$

We require $m_x'(0) = 0$ and in order to obtain the best stress field (closest to yield), we assume

$$m_x = \text{const.} = n_x^2 - 1$$

and thus

$$\alpha(p-1) = 2Rn_x w_1. \tag{50}$$

On the interval $(\xi, 1]$, we have

$$m_x'' = 2\alpha(p-1)$$

which gives by continuity at $x = \xi$

$$m_x = \alpha(p-1)[x^2-1-2\xi x+2\xi] \tag{51}$$

and

$$-\alpha(p-1)(1-\xi)^2 = -1+n_x^2 \tag{52}$$

By equations (48) and (52) we may eliminate n_x^2 to obtain

$$\alpha(p-1) \leq \frac{1}{1-\xi+\xi^2/3} \tag{53}$$

and for the central deflection

$$Rw_0 = Rw_1\xi(2-\xi) = \frac{\xi(2-\xi)\alpha(p-1)}{2[1-\alpha(p-1)(1-\xi)^2]^{\frac{1}{2}}}. \tag{54}$$

Equations (53) and (54) serve as a parametric bound on the pressure in terms of the central deflection. The comparison of the approximate to the exact solution is given in Figs. 4 and 6. In general, the bound is close to the actual curve; the membrane state is reached at $\alpha(p-1) = 3$, instead of $\simeq 4{\cdot}75$, but the solution does give an upper bound on the pressure in terms of the central deflection. In general this need not be the case since inequality (15) contains a measure of the entire displacement field.

For comparison, it should be noted that the results obtained by Duczek [16] are coincident with the curve denoted "bound" in Fig. 4. Because the flow rule is not satisfied everywhere by Duczek's strain-rate field, the stress and velocity fields as given in [16] are not suitable candidates for calculating bounds by the principle developed above. Thus, it is not clear why Duczek's results lie above the exact solution obtained here.

CONCLUSION

Geometric changes in a structure may significantly increase its load carrying capacity beyond the limit load predicted by limit analysis. The mathematical problem is highly non-linear and complex and thus numerical and approximate methods are necessary. A bounding principle which can give useful estimates of the post-yield load deflection behavior has been derived herein. The bounding principle gives a safe (upper) bound on the tractions. For the particular example considered here, agreement between the bound and the exact solution was excellent.

It should be noted that the bound is improvable in the sense that the displacement field may be brought closer to the exact one by, say, a power series expansion. The bounding principle is not in itself able to predict instability and at this point it is not clear if one is able to predict a totally stable configuration *a priori*. This would seem to be an area that merits further investigation.

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Абстракт—Даются принцип предельной деформации для конечно деформируемых, жестких, идеально-пластических конструкций, которые проявляют геометрическую устойчивость. Задача представлена выражениями тензора напряжений Кирхгоффа и тензора деформации Грина, причем используется принцип виртуальной работы и постулат Дракера.

Используется принцип для определения предела поведения цилиндрической оболочки, подверженной вунтреннему давлению. Предел сравнивается с точными результатами, также полученными в предложенной работе.